Last class:

surface area

parametrized surface S given by $\Phi : D \to \mathbb{R}^3$, $D \subset \mathbb{R}^2$

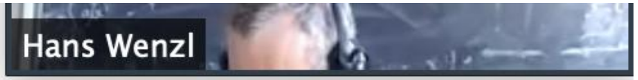
got tangent vectors

$$T_u = \frac{\partial \Phi}{\partial u}, \qquad T_v = \frac{\partial \Phi}{\partial v}$$

area of parallelogram spanned by T_u and T_v

$$= \| T_u \times T_v \|$$

area of S = $\iint_D \| T_u \times T_v \| \, du\, dv$



Special case:

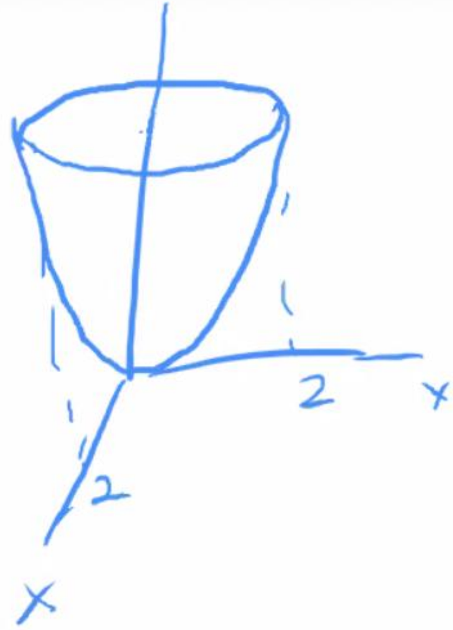
S graph of a function

$$z = f(x,y)$$

example paraboloid $\quad z = x^2+y^2$

$z \leq 4$

$$f(x,y) = x^2+y^2$$



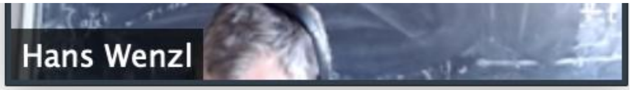
natural parametrization

$$x = u \,,\quad y = v\,,\quad z = f(u,v) = u^2+v^2$$

Remark: Often does not bother with using u and v
just use $z = x^2+y^2$

showed $\|T_u \times T_v\| = \sqrt{\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2+1}$

or $\|T_x \times T_y\| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}$



$$f(u,v) = u^2 + v^2$$

$$\frac{\partial f}{\partial u} = 2u \qquad \frac{\partial f}{\partial v} = 2v$$

$$\| T_u \times T_v \| = \sqrt{4u^2 + 4v^2 + 1}$$

region D: $0 \le z \le 4$

$$0 \le u^2 + v^2 \le 4$$

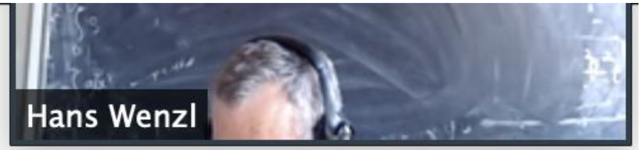
D = disk of radius 2

$$\text{area} = \iint_D \sqrt{4u^2 + 4v^2 + 1}\, du\, dv$$

use polar coordinates:

$$\boxed{\begin{array}{c} u = r\cos\theta \\ v = r\sin\theta \\ 0 \le r \le 2, \quad 0 \le \theta \le 2\pi \end{array}}$$

$$\sqrt{4u^2 + 4v^2 + 1} = \sqrt{4r^2 + 1}$$



$$\text{area} = \int_0^{2\pi}\int_0^{2} \sqrt{4r^2+1}\; r\,dr\,d\theta$$

polar coordinates

substitute
$$\boxed{\begin{aligned} w &= 4r^2+1 \\ \frac{dw}{dr} &= 8r \end{aligned}} \longrightarrow dw = 8r\,dr$$

$$\int \sqrt{4r^2+1}\,dr \rightsquigarrow \int \sqrt{w}\,\frac{1}{8}\,dw = \frac{1}{8}\,\frac{2}{3}\,w^{3/2} + C$$

$$\Rightarrow \text{area} = \int_0^{2\pi} \frac{1}{12}\left(4r^2+1\right)^{3/2}\Big|_0^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{12}\left(17^{3/2} - 1^{3/2}\right) d\theta = \boxed{\frac{2\pi}{12}\left(17^{3/2}-1\right)}$$

$\vdots$

7.5 Integrating functions on a Surface

given: parametrized surface S $\quad \Phi : D \to \mathbb{R}^3$

function $\quad f : \mathbb{R}^3 \to \mathbb{R}$

(or just $f : S \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$)

__Def.__

$$\iint_S f \, dS = \iint_D f(\Phi(u,v)) \, \|T_u \times T_v\| \, du \, dv$$

__Remark:__ If $f(x,y,z) = 1 \quad \forall \, (x,y,z) \in S$

we get the area of S

Examples

(1) S upper hemisphere of radius 1

$$(x^2+y^2+z^2=1, \; z \geq 0)$$

$$f(x,y,z) = z$$

Calculate $\iint_S f \, dS = \iint_S z \, dS$

Solution: need parametrization of S

spherical coordinates: radius $=1 \Rightarrow \rho = 1$

$$x = \sin\phi \cos\theta$$
$$y = \sin\phi \sin\theta$$
$$z = \cos\phi$$

$$\Phi(\phi, \theta) = (\sin\phi \cos\theta, \ \sin\phi \sin \ldots$$

$$\boxed{\begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \frac{\pi}{2} \end{array}} \leftarrow \text{only upper hemisphere } z \ge 0$$

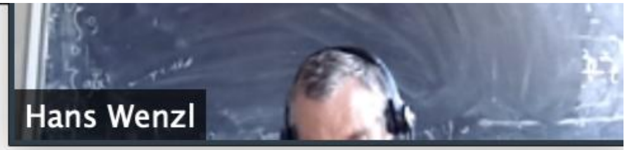
$$T_\theta = \frac{\partial \Phi}{\partial \theta} = (-\sin\phi \sin\theta, \ \sin\phi \cos\theta, \ 0)$$

$$T_\phi = \frac{\partial \Phi}{\partial \phi} = (\cos\phi \cos\theta, \ \cos\phi \sin\theta, \ -\sin\phi)$$

need to calculate

$$T_\theta \times T_\phi = -\sin\phi \, (\sin\phi \cos\theta, \ \sin\phi \sin\theta, \ \cos\theta)$$

$$\|T_\theta \times T_\phi\| = |\sin\phi| \sqrt{\ldots\ldots} = \sin\phi \underset{\text{for } 0 \le \phi \le \pi}{\ge 0}$$



again: $\| T_\theta \times T_\phi \| = \sin\phi$

aside: for general radius ρ

$$\| T_\phi \times T_\theta \| = \rho^2 \sin\phi$$

$$\Rightarrow \iint_S f \, dS = \int_0^{2\pi}\int_0^{\pi/2} \underbrace{f(\Phi(\phi,\theta))}_{\cos\phi} \underbrace{\| T_\theta \times T_\phi \|}_{\underbrace{\sin\phi}_{\rho=1}} \, d\phi \, d\theta$$

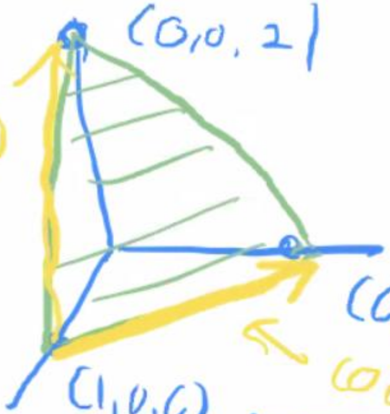
$f(x,y,z) = z = \cos\phi$

$$= \int_0^{2\pi}\int_0^{\pi/2} \cos\phi \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\sin^2\phi \Big|_0^{\pi/2} d\theta = \int_0^{2\pi} \frac{1}{2}(1-0)\, d\theta = \pi$$

② Let S be triangle with corners $(1,0,0)$, $(0,1,0)$ and $(0,0,2)$

Calculate $\iint_S y \, dS$ i.e. $f(x,y,z) = y$

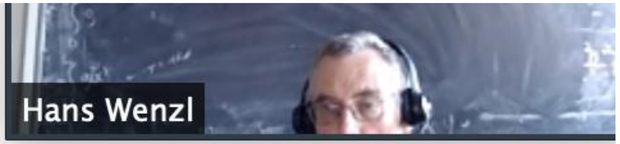


triangle lies on plane given by $\boxed{2x+2y+z=2}$
(check that corners satisfy equation)

or find normal vector $\vec{n} = (-1,1,0) \times (-1,0,2)$
$= (2,2,1)$

equ. of plane: $(2,2,1)\cdot(x-1,y,z)=0$

$\Rightarrow \boxed{z = 2-2x-2y}$ possible values for x and y
$0 \le x \le 1$, $0 \le y \le x$



again D given by $0 \le x \le 1$

$0 \le y \le x$

(given triangle projected onto xy plane.

$$\boxed{z = 2 - 2x - 2y = g(x,y)}$$

use formula for graph:

$$\|T_x \times T_y\| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$$

$$\frac{\partial g}{\partial x} = -2 \qquad \frac{\partial g}{\partial y} = -2$$

$$= \sqrt{(-2)^2 + (-2)^2 + 1} = \sqrt{9} = 3$$

solution: $\iint y \, dS = \int_0^1 \int_0^x y \, \|T_x \times T_y\| \, dy \, dx$

$$= \int_0^1 \int_0^x 3y \, dy \, dx = \frac{1}{2}$$